

# Generators of projective MV-algebras

Francesco Lacava · Donato Saeli

**ABSTRACT** In the last decade, interest in projective MV-algebras has grown greatly; see [1], [5] e [6]. In this paper we establish a necessary and sufficient condition for  $n$  elements of the free  $n$ -generator MV-algebra to generate a projective MV-algebra. This generalizes the characterization of the  $n$  free generators proved in [7]. Using this, some classes of projective generators for bigenerated MV-algebras, are given. In particular, some effective procedures to determine, by elementary methods, generators of projective MV-algebras are explained.

Nell'ultimo decennio, l'interesse per le MV-algebre proiettive è cresciuto sensibilmente; cfr. [1], [5] e [6]. In questa nota viene dimostrata una condizione necessaria e sufficiente perchè  $n$  elementi di  $Free_n$  siano generatori di MV-algebre proiettive, che generalizza la caratterizzazione degli  $n$  generatori liberi dimostrata in [7]. Utilizzando tale condizione nel caso di MV-algebre bigenerate sono state determinate alcune classi di generatori proiettivi. In particolare vengono esposti alcuni procedimenti effettivi per la determinazione dei generatori di algebre proiettive, con metodi elementari.

**KEYWORDS** MV-algebras · projective · retract · piecewise linear map

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## 1 Introduction

Throughout this paper, by an MV-algebra  $A$  we mean a *semisimple* MV-algebra, that is, a subalgebra of a direct power of the standard MV-algebra  $[0, 1]$ . Thus for some set  $T$ , without loss of generality we can assume that each  $a \in A$  is a  $[0, 1]$ -valued function defined on  $T$ .

In particular, since the free  $n$ -generator MV-algebra  $Free_n$  is semisimple, throughout we will identify  $Free_n$  with the algebra of McNaughton functions on the  $n$ -cube  $[0, 1]^n$ .

An MV-algebra  $A$  is said to be *essentially*  $n$ -generated if it has a generating set of  $n$  elements, but no generating set of  $n - 1$  elements. We refer to [4], for all unexplained notation and terminology.

Throughout this paper, whenever  $a$  generates  $A$  we will assume that  $a(0) = 0$ .

Let  $L_{n+1}$  denote the Łukasiewicz MV-chain with  $n+1$  elements. Then following [7], we will introduce the partial map  $t : L_{n+1} \rightarrow L_{n+1}$  by stipulating that, for each  $a \in L_{n+1}$

$$t(a) = \begin{cases} (ra)' & \text{if there is } r \in \mathbb{Z}, r > 0, \\ & \text{such that } ra < 1 \text{ and } (r+1)a = 1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In particular, the map  $t$  is undefined for  $a = 0, 1$ . We further define

$$\begin{aligned} t^0(a) &= a, \\ t^{s+1}(a) &= t(t^s(a)). \end{aligned}$$

DEFINITION. - An element  $a \in L_{n+1}$  is said to be a *cyclic generator* of  $L_{n+1}$ , if there is an integer  $k \geq 0$  such that  $t^k(a) = 1/n$ .

PROPOSITION 1.1. - If  $p$  is a prime number, then every element  $a \in L_{p+1}$ ,  $a \neq 0, 1$ , is a cyclic generator of  $L_{p+1}$ . This is [7, 2.4].

For  $p$  a prime number, let the integers  $m$  and  $p$  be such that  $0 < m < p$ . Let  $k$  be the smallest positive integer satisfying  $t^k\left(\frac{m}{p}\right) = \frac{1}{p}$ .

Let the sequence of elements of  $L_{p+1}$ , be given by

$$t\left(\frac{m}{p}\right) = \left(n_1 \frac{m}{p}\right)', t^2\left(\frac{m}{p}\right) = \left(n_2 t\left(\frac{m}{p}\right)\right)', \dots, t^k\left(\frac{m}{p}\right) = \left(n_k t^{k-1}\left(\frac{m}{p}\right)\right)'$$

Then the term  $\gamma_{m,p}$  in the variable  $x$  is defined by

$$\gamma_{m,p}(x) = \left(n_k(n_{k-1} \dots (n_2(n_1 x)')' \dots )'\right)'.$$

When interpreted in the  $Free_1$  algebra, the term  $\gamma_{m,p}$  represents a McNaughton function  $g_{m,p}$  whose graph has three linear pieces, two of which (eventually degenerate into points) are horizontal, respectively at level 0 and at level 1. Moreover,  $g_{m,p}(m/p) = 1/p$  and  $g_{m,p}(x) \neq 1/p$  for each  $x \in [0, 1]$  different from  $m/p$ . This is [7, 2.5].

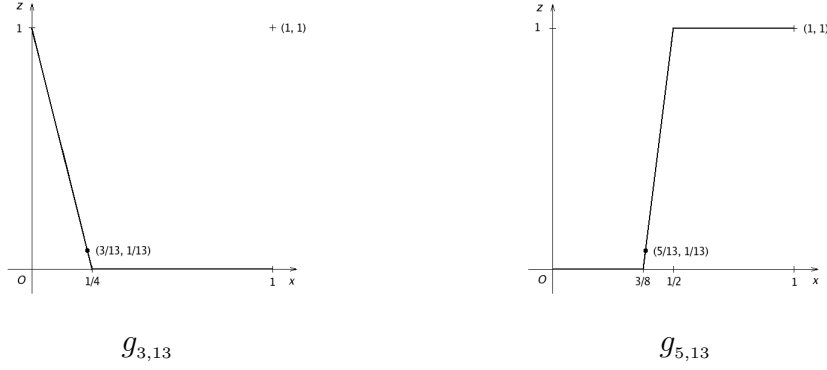


Fig. 1

For  $p$ , a prime number, we can define the term  $\lambda_p$  as follows:

$$\lambda_p(x) = \left( px \wedge p((p-1)x) \right)'$$

If interpreted in the  $Free_1$  algebra, the term  $\lambda_p$  represents a McNaughton function  $l_p$  such that  $l_p(x) = 0$  if and only if  $x = 1/p$ . The last nontrivial statement follows by noting that  $x > 1/p$  implies  $p[1 - (p-1)x] < 1$ .

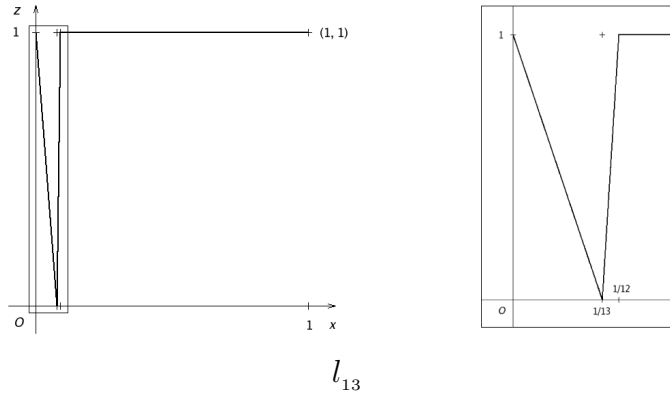


Fig. 2

For all integers  $0 < m < p$ , with  $p$  a prime number, let

$$\eta_{m,p}(x) = \lambda_p(\gamma_{m,p}(x)).$$

If interpreted in the  $Free_1$  algebra, the term  $\eta_{m,p}$  represents a McNaughton function  $t_{m,p}$  such that:

- (1.1) For all  $x \in [0, 1]$ ,  $t_{m,p}(x) = 0$  if and only if  $x = m/p$ .
- (1.2) For every  $L \in \mathbb{Z}$ ,  $L > 0$  there is  $Q = Q(L) \in \mathbb{Z}$ ,  $Q > 0$  such that, for all  $x \in [0, 1]$  with  $|x - m/p| \geq 1/L$ , we have  $t_{m,p}(x) \geq 1/Q$ .

This is [7, 2.6].

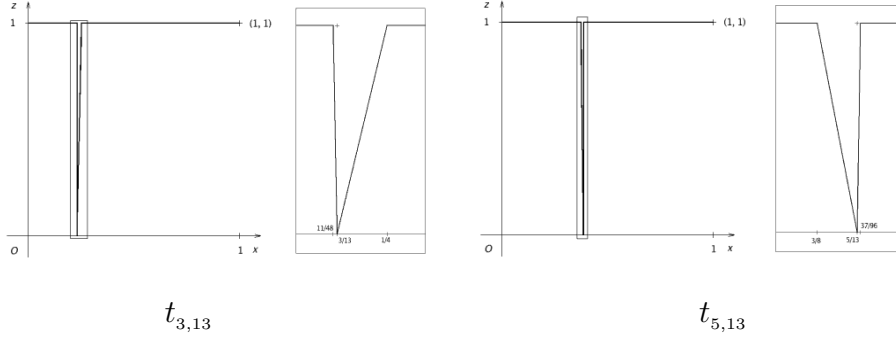


Fig. 3

Following [4, 6.2.3], we say that an element  $a$  of an MV-algebra is archimedean if there is an integer  $n \geq 1$  such that  $na + na = na$ .

LEMMA 1.2. - Suppose the elements  $a_1, a_2, \dots, a_n$  (essentially) generate the MV-algebra  $A$  of  $[0, 1]$ -valued functions. Let  $m_i < p_i$  be integers  $> 0$  with each  $p_i$  prime ( $1 \leq i \leq n$ ). Then the element

$$\bigvee_{i=1}^n \eta_{m_i, p_i}(a_i)$$

fails to be archimedean if and only if for every  $\varepsilon > 0$  there is  $\bar{x} \in [0, 1]^n$  such that for  $i = 1, \dots, n$ , we have  $\left| a_i(\bar{x}) - \frac{m_i}{p_i} \right| < \varepsilon$ .

PROOF. - This follows at once from point (1.1) aforesaid. ▲

PROPOSITION 1.3. - Suppose the MV-algebras  $A$  and  $B$  of  $[0, 1]$ -valued functions are essentially generated by sets  $\{a_1, a_2, \dots, a_n\}$  and  $\{b_1, b_2, \dots, b_n\}$  respectively. Then the following two conditions are equivalent:

- The map  $f : a_i \mapsto b_i$  is (uniquely) extendible to an isomorphism of  $A$  onto  $B$
- For all integers  $0 < m_i < p_i$ , with each  $p_i$  prime, the element  $\bigvee_{i=1}^n \eta_{m_i, p_i}(a_i)$  is archimedean if and only if so is the element  $\bigvee_{i=1}^n \eta_{m_i, p_i}(b_i)$ .

PROOF. - ( $\Rightarrow$ ) The statement is obvious.

( $\Leftarrow$ ) Vice versa, by contradiction, if we suppose that  $f$  cannot be extended to an isomorphism between  $A$  and  $B$ , then a term  $t$  exists so that

$$t(a_1, a_2, \dots, a_n) = \hat{0} \text{ and } t(b_1, b_2, \dots, b_n) \neq \hat{0}.$$

It follows that there are  $\mathbf{x}_0 \in [0, 1]^n$  and  $\delta > 0$  such that for every  $\mathbf{x} \in \mathcal{I}(\mathbf{x}_0, \delta)$  (the  $\delta$ -neighbourhood of  $\mathbf{x}_0$ ) we have

$$t(b_1(\mathbf{x}), b_2(\mathbf{x}), \dots, b_n(\mathbf{x})) \neq 0.$$

Since the  $b_i(\mathbf{x})$  are McNaughton functions, it is possible to choose a  $\mathbf{x}_0$  such that a neighborhood of  $(b_1(\mathbf{x}_0), b_2(\mathbf{x}_0), \dots, b_n(\mathbf{x}_0))$  is contained in the image by  $\mathbf{b}(\mathbf{x}) = (b_1(\mathbf{x}), b_2(\mathbf{x}), \dots, b_n(\mathbf{x}))$  of  $\mathcal{I}(\mathbf{x}_0, \delta)$ .

Furthermore there are  $\bar{\mathbf{x}} \in [0, 1]^n \cap \mathbb{Q}$  and  $\bar{\delta} > 0$  such that for every

$\mathbf{x} \in \mathcal{J}(\bar{\mathbf{x}}, \bar{\delta})$  we have  $t(b_1(\mathbf{x}), b_2(\mathbf{x}), \dots, b_n(\mathbf{x})) \neq 0$ , also  $b_i(\bar{\mathbf{x}}) = \frac{m_i}{p_i}$  with

$p_i$  prime numbers. So clearly, the element  $\bigvee_{i=1}^n \eta_{m_i, p_i}(b_i)$  is not archimedean

and, by hypothesis,  $\bigvee_{i=1}^n \eta_{m_i, p_i}(a_i)$  is not archimedean either.

Now by the lemma 1.2 there is a  $\bar{\varepsilon}$  such that, for each  $\mathbf{y}$  satisfying

$$\left| y_i - \frac{m_i}{p_i} \right| < \bar{\varepsilon} \quad (i = 1, \dots, n), \quad \text{we have } t(\mathbf{y}) \neq 0.$$

Since there is also an  $\mathbf{x} \in [0, 1]^n$  with  $\left| a(x_i) - \frac{m_i}{p_i} \right| < \bar{\varepsilon} \quad (i = 1, \dots, n)$ ,

we conclude  $t(a_1(\mathbf{x}), a_2(\mathbf{x}), \dots, a_n(\mathbf{x})) \neq 0$ , which is a contradiction.  $\blacktriangle$

In particular, the previous proposition tells us that in the free one-generator MV-algebra  $Free_1$ , two subalgebras  $A$  and  $B$ , generated respectively by  $a$  and  $b$ , are isomorphic if and only if  $\max(a) = \max(b) = l$ , that is if and only if, whatever are  $0 < m < p$ , with  $p$  prime number and  $\frac{m}{p} \leq l$ , we have that  $\eta_{m,p}(a)$  is archimedean if and only if  $\eta_{m,p}(b)$  is archimedean.

Let  $f, g \in Free_1$ , with  $f(0) = g(0) = 0$ . Then there is a sequence  $a_0 = 0 < a_1 < \dots < a_{k-1} < a_k = 1$  of rational numbers in  $[0, 1]$ , together with linear functions with integer coefficients  $p_1, \dots, p_k, q_1, \dots, q_k : \mathbb{R} \rightarrow \mathbb{R}$  such that:

- 1) over each interval  $[a_{i-1}, a_i]$ ,  $f$  coincides with  $p_i$ , and  $g$  coincides with  $q_i$ , ( $i = 1, \dots, k$ )
- 2) for each  $j = 2, \dots, k$ , either  $p_{j-1}$  is distinct from  $p_j$  or else  $q_{j-1}$  is distinct from  $q_j$ .

Let the function  $c = (f, g) : [0, 1] \rightarrow [0, 1]^2$  defined by  $c(t) = (f(t), g(t))$ ; the shape of range of  $(f, g)$  is of course the broken line in  $[0, 1]^2$  joining, in the order, the points

$$P_0 \equiv (f(a_0), g(a_0)), P_1 \equiv (f(a_1), g(a_1)), \dots, P_k \equiv (f(a_k), g(a_k)).$$

We will name these points, that are known as the nodes of the range of  $(f, g)$ , *extremals* of the pair  $f$  and  $g$ .

**PROPOSITION 1.4.** - *If  $f, g$  and  $f_1, g_1$  are two pairs of  $Free_1$  elements such that the range of  $(f, g)$  coincides with the range of  $(f_1, g_1)$ , then the algebra generated by  $f, g$  is isomorphic to the algebra generated by  $f_1, g_1$ .*

**PROOF.** - It follows straight from proposition 1.3.  $\blacktriangle$

EXAMPLE 1.5. - The functions

$$f(x) = \begin{cases} 6x & \text{for } 0 \leq x \leq \frac{1}{6} \\ 3 - 12x & \text{for } \frac{1}{6} \leq x \leq \frac{1}{4} \\ 12x - 3 & \text{for } \frac{1}{4} \leq x \leq \frac{1}{3} \\ 1 & \text{for } \frac{1}{3} \leq x \leq 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 4x & \text{for } 0 \leq x \leq \frac{1}{6} \\ 2 - 8x & \text{for } \frac{1}{6} \leq x \leq \frac{1}{4} \\ 8x - 2 & \text{for } \frac{1}{4} \leq x \leq \frac{3}{8} \\ 1 & \text{for } \frac{3}{8} \leq x \leq 1 \end{cases}$$

and the functions

$$f_1(x) = \begin{cases} 3x & \text{for } 0 \leq x \leq \frac{1}{3} \\ 1 & \text{for } \frac{1}{3} \leq x \leq 1 \end{cases} \quad \text{and} \quad g_1(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}$$

generate isomorphic algebras.

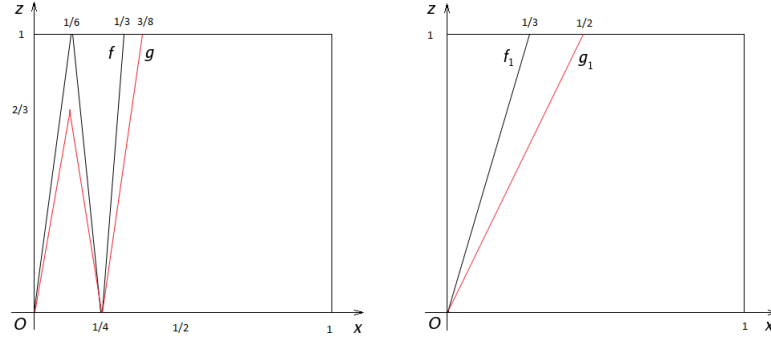


Fig. 4

COROLLARY 1.6. - If  $f, g$  and  $f_1, g_1$  are two pairs of  $Free_1$  elements with the same extremals, then the algebra generated by  $f, g$  is isomorphic to the algebra generated by  $f_1, g_1$ .

## 2 Projective algebras

We recall that:

- A MV-algebra  $A$  is named *projective* if, given any MV-algebras  $B$  and  $C$ , for each epimorphism  $g$  from  $C$  on  $B$  and each omomorphism  $f$  from  $A$  in  $B$ , then an omomorphism  $h$  from  $A$  in  $C$ , such that  $h \circ g = f$ , always exists.
- The MV-algebra  $B$  is said a *retract* of the MV-algebra  $A$ , if there are a monomorphism  $\chi : B \rightarrow A$  and an epimorphism  $\varepsilon : A \rightarrow B$  such that  $\varepsilon\chi : B \rightarrow B$  is the identity; so if  $B$  is a retract of  $A$ , then  $\chi(B)$  is a subalgebra of  $A$  such as the endomorphism  $\chi\varepsilon$  of  $A$  restricted to  $\chi(B)$  is the identity. It is known that a retract of a free MV-algebra is projective.

We try, in this paragraph, to characterize the projective subalgebras of  $Free_n$ . First of all, we observe, that if  $A$  is a projective MV-algebra essentially  $n$ -generated then it is a retract of  $Free_n$ . More generally, if  $A$ ,  $n$ -generated, is a retract of  $Free_m$  with  $m > n$ , then  $A$  is a retract of  $Free_n$ .

LEMMA 2.1. - *If  $x_1, \dots, x_n, u_1, \dots, u_n$  are elements of a MV-algebra,  $t$  is a term and  $\delta(x, u)$  is the Chang's distance, then there is  $k \in \mathbb{N}$  such that*

$$\delta(t(x_1, \dots, x_n), t(u_1, \dots, u_n)) \leq k[\delta(x_1, u_1) + \dots + \delta(x_n, u_n)].$$

PROOF. - By an easy induction from a theorem on elementary properties of the Chang's distance [2, 3.14].  $\blacktriangle$

PROPOSITION 2.2. - *Let  $B$  be a  $n$ -generated subalgebra of the free algebra  $A = Free_n$  and let be  $g_1, \dots, g_n$  the generators of  $A$ . If  $\varphi$  is an epimorphism from  $A$  on  $B$ , then  $\varphi(g_1) = d_1, \dots, \varphi(g_n) = d_n$  generate  $B$  and the kernel of  $\varphi$  is the ideal  $I$  generated by  $\delta(d_1, g_1), \dots, \delta(d_n, g_n)$ .*

PROOF. - If  $a \in A$ , where  $a = t(g_1, \dots, g_n)$ , is such that  $\varphi(a) = 0$ , then:  $a = \delta(a, 0) = \delta(t(g_1, \dots, g_n), t(d_1, \dots, d_n)) \leq k[\delta(g_1, d_1) + \dots + \delta(g_n, d_n)]$ , for an appropriate  $k \in \mathbb{N}$ , that is  $a \in I$ .  $\blacktriangle$

Thus, in order that  $\varphi$  restricted to  $B$  is an injection, it is necessary (and sufficient) that if  $b \in B$  and  $b \leq k[\delta(g_1, d_1) + \dots + \delta(g_n, d_n)]$ , then  $b = 0$ .

Finally with the notations and conditions set above (proposition 2.2), we introduce a set  $K$ , that we say "equalizer" of  $B$ :

DEFINITION 2.3. -  $K = \{\mathbf{x} \in [0, 1]^n : d_1(\mathbf{x}) = g_1(\mathbf{x}), \dots, d_n(\mathbf{x}) = g_n(\mathbf{x})\}$ .

THEOREM 2.4. - *Let  $B$  be a  $n$ -generated subalgebra of  $A = Free_n$ ,  $\varphi$  an epimorphism from  $A$  on  $B$  and let  $g_1, \dots, g_n$  be the generators of  $A$ . Set  $d_1 = \varphi(g_1), \dots, d_n = \varphi(g_n)$ , then the epimorphism  $\varphi$  restricted to  $B$ , is an isomorphism if and only if for every  $\mathbf{u} \in [0, 1]^n$  there is  $\mathbf{x} \in K$  such that*

$$d_1(\mathbf{u}) = d_1(\mathbf{x}), \dots, d_n(\mathbf{u}) = d_n(\mathbf{x}).$$

PROOF. - If for  $b \in B$ ,  $b = t(d_1, \dots, d_n)$ , were  $b \leq k[\delta(d_1, g_1) + \dots + \delta(d_n, g_n)]$  and  $b \neq 0$ , then there would be  $\mathbf{u} \in [0, 1]^n$  such that:  $b(\mathbf{u}) = t(d_1(\mathbf{u}), \dots, d_n(\mathbf{u})) \neq 0$ ; but by hypothesis,

$\mathbf{x}_0 \in K$  exists such that  $d_1(\mathbf{u}) = d_1(\mathbf{x}_0), \dots, d_n(\mathbf{u}) = d_n(\mathbf{x}_0)$

and so  $t(d_1(\mathbf{u}), \dots, d_n(\mathbf{u})) = t(d_1(\mathbf{x}_0), \dots, d_n(\mathbf{x}_0)) = b(\mathbf{x}_0) \neq 0$ .

But, since  $d_1(\mathbf{x}_0) = g_1(\mathbf{x}_0), \dots, d_n(\mathbf{x}_0) = g_n(\mathbf{x}_0)$  we have:

$$(k[\delta(d_1, g_1) + \dots + \delta(d_n, g_n)])(\mathbf{x}_0) =$$

$$k[\delta(d_1(\mathbf{x}_0), g_1(\mathbf{x}_0)) + \dots + \delta(d_n(\mathbf{x}_0), g_n(\mathbf{x}_0))] = 0$$

and so would be  $0 \neq b(\mathbf{x}_0) \leq (k[\delta(d_1, g_1) + \dots + \delta(d_n, g_n)])(\mathbf{x}_0) = 0$ .

Vice versa, let  $\mathbf{u}_1 \in [0, 1]^n$  be such that for every  $\mathbf{x} \in K$ ,

$$d_i(\mathbf{u}_1) \neq d_i(\mathbf{x}) \quad \text{for some } i, 1 \leq i \leq n,$$

consequently there is  $\varepsilon > 0$  such as for every  $\mathbf{v} \in \mathcal{J}(\mathbf{u}_1, \varepsilon)$  and  $\mathbf{x} \in K$  it is  $d_i(\mathbf{v}) \neq d_i(\mathbf{x})$ . In particular there are integers  $0 < m_1 < p_1, \dots, 0 < m_n < p_n$ , with  $p_1, \dots, p_n$  prime numbers,  $\mathbf{u}_2 \in \mathcal{J}(\mathbf{u}_1, \varepsilon)$  and  $\varrho > 0$  such that:

$$d_1(\mathbf{u}_2) = \frac{m_1}{p_1}, \dots, d_n(\mathbf{u}_2) = \frac{m_n}{p_n} \text{ and } \mathcal{J}(\mathbf{u}_2, \varrho) \subseteq \mathcal{J}(\mathbf{u}_1, \varepsilon).$$

Then we can choose  $h \in \mathbb{N}$  so that the element

$$b = \left( h \bigvee_{i=1}^n \eta_{m_i, p_i}(d_i) \right)'$$

is nonzero only in  $\mathcal{J}(\mathbf{u}_2, \varrho)$ . Thus, for  $k \in \mathbb{N}$  large enough, is  $b \leq k[\delta(g_1, d_1) + \dots + \delta(g_n, d_n)]$ , but  $b \neq 0$  also. ▲

**COROLLARY 2.5.** - *Let  $A$  be a  $n$ -generated MV-algebra.  $A$  is projective if and only if is isomorphic to a subalgebra  $B$  of  $\text{Free}_n$ , whereby, however we choose  $\mathbf{u} \in [0, 1]^n$  there is  $\mathbf{x} \in K$  such that*

$$d_1(\mathbf{u}) = d_1(\mathbf{x}), \dots, d_n(\mathbf{u}) = d_n(\mathbf{x}).$$

**COROLLARY 2.6 (DI NOLA).** - *Every monogenerated subalgebra of  $\text{Free}_1$  is projective.*

We devote the next section to some constructive applications of theorem 2.4 for bigenerated projective algebras.

### 3 Bigenerated algebras

Since the equalizer  $K$  must be a finite union of triangular simplexes  $T_1, T_2, \dots, T_k$  and each triangular simplex  $T_i$  is determinated by a system

$$\begin{cases} a_i x + b_i y + c_i \geq 0 \\ a_{i1}x + b_{i1}y + c_{i1} \geq 0 \\ a_{i2}x + b_{i2}y + c_{i2} \geq 0 \end{cases} \quad (\alpha_i)$$

of three linear inequalities in two variables, so we can state, according to previous observations:

**THEOREM 3.1.** - *A subalgebra  $B$  of  $\text{Free}_2$  generated by  $a$  and  $b$ , is projective if and only if the pairs  $(\frac{m_1}{p_1}, \frac{m_2}{p_2})$ , with  $0 < m_j < p_j$  and  $p_j$  prime ( $j = 1, 2$ ), that satisfy one of the systems  $(\alpha_1), (\alpha_2), \dots, (\alpha_k)$ , are all and only those for wich the element  $\eta_{m_1, p_1}(a) \vee \eta_{m_2, p_2}(b)$  is not archimedean.*

**PROOF.** - By corollary 2.5. ▲

Therefore, it is important to establish wich are the triangular simplexes that give rise the projective algebras. Certainly, the union of such simplexes must be connected and it must contain  $\mathbf{0}$ . But it can be not convex, as it is easy to see.



Now, we point out a geometrical fact quite elementary, but basic for the following.

REMARK 3.2. - On the plane  $Oxz$ , we consider the points

$$O \equiv (0, 0), P \equiv (a, h), K \equiv (0, k), Q \equiv (a, l),$$

$$O' \equiv (b, 0), P' \equiv (c, h), K' \equiv (b, k) \text{ and } Q' \equiv (c, l),$$

where the real numbers  $a$ ,  $b$  and  $c$ , each nonzero, are pairwise different.

Similarly,  $h$ ,  $k$  and  $l$  are supposed to be all nonzero and pairwise different.

If  $s$  and  $t \in \mathbb{R}$ , the line  $x = s$  intersects the line through  $OP$  and the line through  $KQ$  respectively at points  $S \equiv (s, u)$  and  $S' \equiv (s, u')$ ; and

the line  $x = t$  intersects the line through  $O'P'$  and the line through  $K'Q'$  respectively at points  $T \equiv (t, v)$  and  $T' \equiv (t, v')$ .

respectively at points  $T \equiv (t, v)$  and  $T' \equiv (t, v')$ .

*If, for a given  $s$ , we choose  $t = b + [s(c - b)/a]$  (or if, for a given  $t$ , we choose  $s = a(t - b)/(c - b)$ ), then  $u = v$  and  $u' = v'$ .*

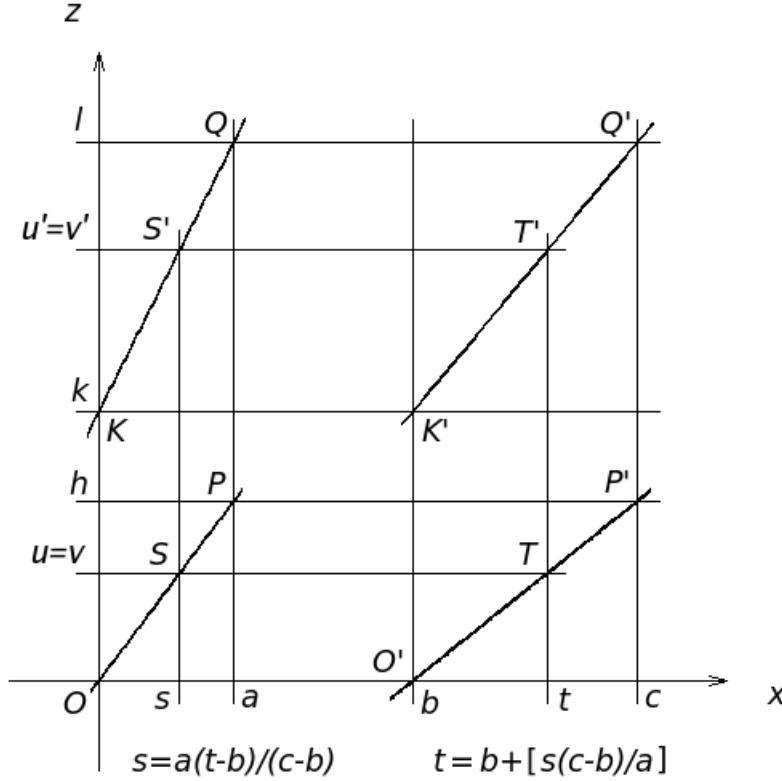


Fig. 5

In the space  $\mathbb{R}^3$ , we denote by  $\Delta_{\mathcal{U}}$  the triangle whose vertices belong to the set  $\mathcal{U} = \{(t_1, v_1, z_1), (t_2, v_2, z_2), (t_3, v_3, z_3)\}$ , where the points  $(t_1, v_1, 0), (t_2, v_2, 0), (t_3, v_3, 0)$  are not aligned. We denote by  $z = f_{\mathcal{U}}(t, v)$  the linear function corresponding to the plane settled by the points of  $\mathcal{U}$ .

**PROPOSITION 3.3.** *Let the sets:*

$$\begin{aligned}\mathcal{A} &= \{(x_1, y_1, a_1), (x_2, y_2, a_2), (x_3, y_3, a_3)\}, \\ \mathcal{B} &= \{(x_1, y_1, b_1), (x_2, y_2, b_2), (x_3, y_3, b_3)\}, \\ \mathcal{C} &= \{(x_1, y_1, 0), (x_2, y_2, 0), (x_3, y_3, 0)\},\end{aligned}$$

*with the condition that the points of  $\mathcal{C}$  are not aligned.*

*Arbitrarily chosen*

*a set  $\mathcal{J} = \{(\xi_1, \eta_1, 0), (\xi_2, \eta_2, 0), (\xi_3, \eta_3, 0)\}$ , whose points are not aligned, a 3-list  $(i_1, i_2, i_3)$  in  $\{1, 2, 3\}$ ,*

*and consequently also the sets:*

$$\begin{aligned}\mathcal{L} &= \{(\xi_1, \eta_1, a_{i_1}), (\xi_2, \eta_2, a_{i_2}), (\xi_3, \eta_3, a_{i_3})\} \text{ and} \\ \mathcal{M} &= \{(\xi_1, \eta_1, b_{i_1}), (\xi_2, \eta_2, b_{i_2}), (\xi_3, \eta_3, b_{i_3})\}.\end{aligned}$$

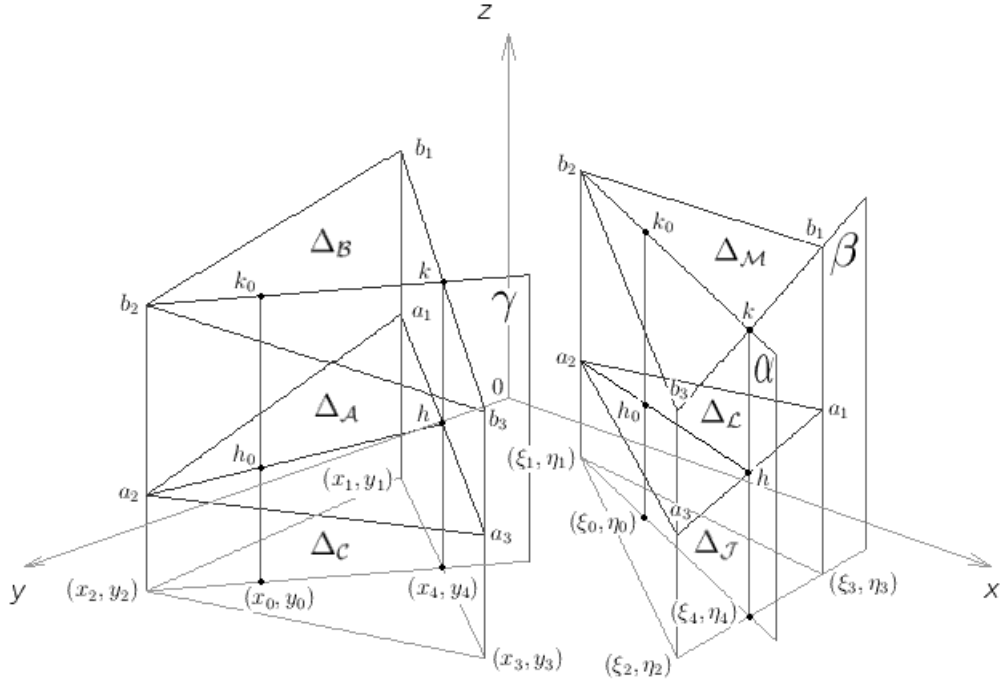
*Then, for every  $(\xi_0, \eta_0, 0) \in \Delta_{\mathcal{J}}$  there is  $(x_0, y_0, 0) \in \Delta_{\mathcal{C}}$  such that*

$$f_{\mathcal{L}}(\xi_0, \eta_0) = f_{\mathcal{A}}(x_0, y_0) \quad \text{and} \quad f_{\mathcal{M}}(\xi_0, \eta_0) = f_{\mathcal{B}}(x_0, y_0).$$

**PROOF.** - Suppose first, that  $(i_1, i_2, i_3)$  is a permutation of  $\{1, 2, 3\}$ .

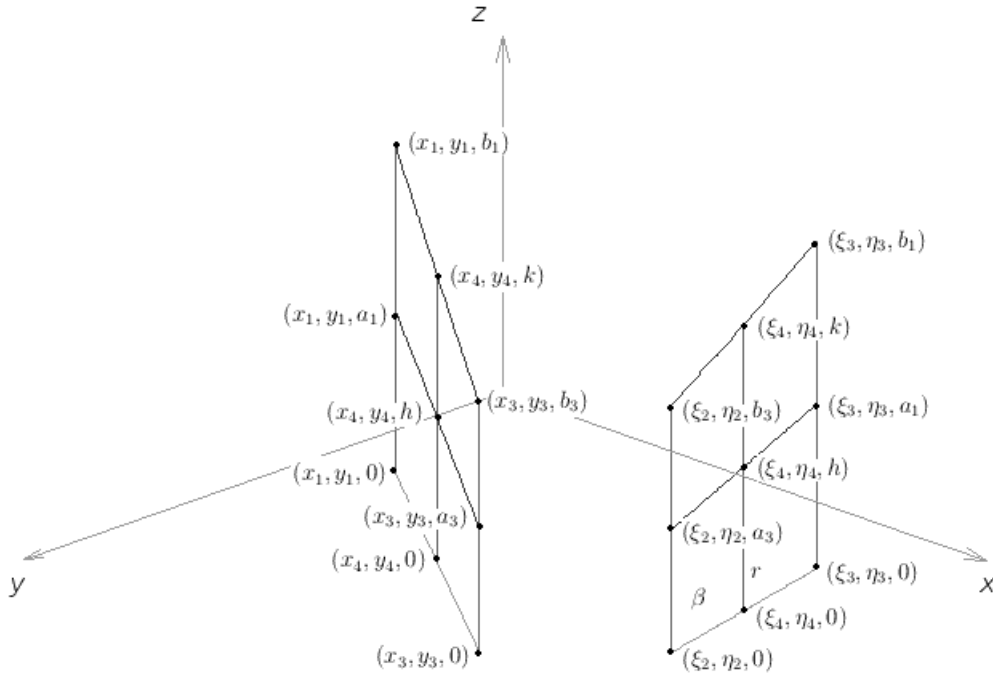
Let  $\alpha$  the plane passing through the points  $(\xi_1, \eta_1, a_{i_1}), (\xi_1, \eta_1, b_{i_1})$  and  $(\xi_0, \eta_0, 0)$ . This plane meets the plane  $\beta$ , passing through the points  $(\xi_2, \eta_2, a_{i_2}), (\xi_2, \eta_2, b_{i_2})$  and  $(\xi_3, \eta_3, a_{i_3})$ , along a straight line  $r$  which intersects the side of the triangle  $\Delta_{\mathcal{L}}$ , of extremes  $(\xi_2, \eta_2, a_{i_2})$  and  $(\xi_3, \eta_3, a_{i_3})$  and the side of the triangle  $\Delta_{\mathcal{M}}$ , of extremes  $(\xi_2, \eta_2, b_{i_2})$  and  $(\xi_3, \eta_3, b_{i_3})$ , respectively, in points  $(\xi_4, \eta_4, h)$  and  $(\xi_4, \eta_4, k)$  (Fig. 6.0). For the remark 3.2, on the side of the triangle  $\Delta_{\mathcal{C}}$ , of extremes  $(x_{i_2}, y_{i_2}, 0)$  and  $(x_{i_3}, y_{i_3}, 0)$ , there is a point  $(x_4, y_4, 0)$  such that  $f_{\mathcal{A}}(x_4, y_4) = h$  and  $f_{\mathcal{B}}(x_4, y_4) = k$  (Fig 6.1). Just now consider the plane  $\gamma$  passing through the points  $(x_{i_1}, y_{i_1}, a_{i_1}), (x_{i_1}, y_{i_1}, b_{i_1})$  e  $(x_4, y_4, 0)$  and the assertion follows by a further application of the remark 3.2 (Fig. 6.2).

If  $(i_1, i_2, i_3)$  is not a permutation of  $\{1, 2, 3\}$ , similar proofs hold.  $\blacktriangle$



Here the 3-list is (2,3,1). For simplicity, points in plane  $z = 0$  are denoted with the first two coordinates, while the other points only with their heights.

Fig. 6.0



First application of remark 3.2, this yields the point  $(x_4, y_4, 0)$  corresponding to the point  $(\xi_4, \eta_4, 0)$

Fig. 6.1

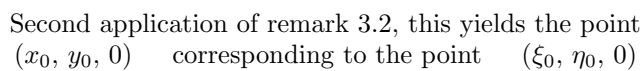


Fig. 6.2

i) - A first very simple case arises if the equalizer is:

$$K = \{(x, y) \in [0, 1]^2 : g_1(y) \leq x \leq g_2(y) \text{ and } f_1(x) \leq y \leq f_2(x)\},$$

where  $f_1(x)$ ,  $f_2(x)$ ,  $g_1(y)$  and  $g_2(y)$  are McNaughton functions and the follow conditions are satisfied:

$$\begin{aligned} f_1(x) &\leq f_2(x), \quad g_1(y) \leq g_2(y); \\ y &\geq f_1(g_1(y)), \quad y \leq f_2(g_1(y)), \quad y \leq f_2(g_2(y)), \quad y \geq f_1(g_2(y)). \end{aligned} \quad (\bullet)$$

Set :

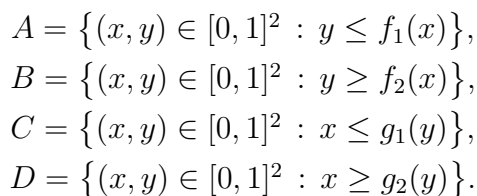


Fig. 7

The functions

$$d_1(x, y) = \begin{cases} x & \text{if } (x, y) \in A \cup K \cup B, \\ g_1(y) & \text{'' } (x, y) \in C, \\ g_2(y) & \text{'' } (x, y) \in D \end{cases}$$

$$d_2(x, y) = \begin{cases} y & \text{if } (x, y) \in C \cup K \cup D, \\ g_1(y) & \text{'' } (x, y) \in A, \\ g_2(y) & \text{'' } (x, y) \in B \end{cases}$$

generate, by the corollary 2.5 and the proposition 3.3, a projective subalgebra of  $Free_2$ .

EXAMPLE 3.4. - The McNaughton functions

$$f_1(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \frac{1}{3} \\ 1 - 2x & \text{for } \frac{1}{3} \leq x \leq \frac{1}{2} \\ 2x - 1 & \text{for } \frac{1}{2} \leq x \leq \frac{2}{3} \\ 1 - x & \text{for } \frac{2}{3} \leq x \leq 1 \end{cases}, \quad f_2(x) = \begin{cases} 1 - x & \text{for } 0 \leq x \leq \frac{1}{2} \\ x & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases};$$

$$g_1(y) = \begin{cases} 0 & \text{for } 0 \leq y \leq \frac{1}{4} \\ 4y - 1 & \text{for } \frac{1}{4} \leq y \leq \frac{1}{3} \\ 1 - 2y & \text{for } \frac{1}{3} \leq y \leq \frac{1}{2} \\ 2y - 1 & \text{for } \frac{1}{2} \leq y \leq \frac{2}{3} \\ 1 - y & \text{for } \frac{2}{3} \leq y \leq 1 \end{cases}, \quad g_2(y) = \begin{cases} 1 & \text{for } 0 \leq y \leq \frac{1}{3} \\ 2 - 3y & \text{for } \frac{1}{3} \leq y \leq \frac{2}{5} \\ 2y & \text{for } \frac{2}{5} \leq y \leq \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \leq y \leq \frac{2}{3} \\ 3 - 3y & \text{for } \frac{2}{3} \leq y \leq \frac{5}{7} \\ 4y - 2 & \text{for } \frac{5}{7} \leq y \leq \frac{3}{4} \\ 1 & \text{for } \frac{3}{4} \leq y \leq 1 \end{cases}$$

satisfy the conditions  $(\bullet)$ ;

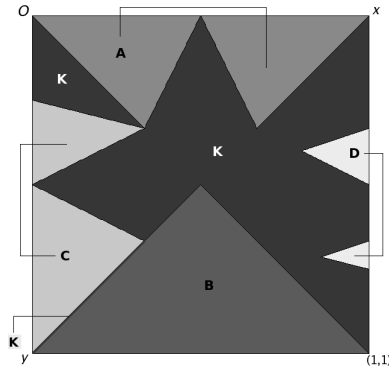


Fig. 8

Therefore

$$d_1(x, y) = \begin{cases} 4y - 1 & \text{if } x \leq 4y - 1 \text{ and } y \leq 1/3, \\ 1 - 2y & \text{'' } x \leq 1 - 2y \text{ and } y \geq 1/3, \\ 2y - 1 & \text{'' } x \leq 2y - 1 \text{ and } y \leq 2/3, \\ 1 - y & \text{'' } x \leq 1 - y \text{ and } y \geq 2/3, \\ 2 - 3y & \text{'' } x \geq 2 - 3y \text{ and } y \leq 2/5, \\ 2y & \text{'' } x \geq 2y \text{ and } y \geq 2/5, \\ 3 - 3y & \text{'' } x \geq 3 - 3y \text{ and } y \leq 5/7, \\ 4y - 2 & \text{'' } x \geq 4y - 2 \text{ and } y \geq 5/7, \\ x & \text{otherwise} \end{cases}$$

and

$$d_2(x, y) = \begin{cases} x & \text{if } y \leq x \text{ and } x \leq 1/3, \\ 1 - 2x & \text{'' } y \leq 1 - 2x \text{ and } x \geq 1/3, \\ 2x - 1 & \text{'' } y \leq 2x - 1 \text{ and } x \leq 2/3, \\ 1 - x & \text{'' } y \leq 1 - x \text{ and } x \geq 2/3, \\ 1 - x & \text{'' } y \geq 1 - x \text{ and } x \leq 1/2, \\ x & \text{'' } y \geq x \text{ and } x \geq 1/2, \\ y & \text{otherwise} \end{cases}$$

are the generators of a projective subalgebra of  $Free_2$ .

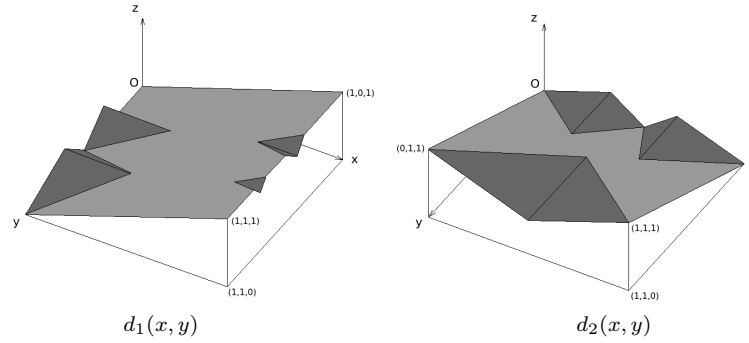


Fig. 9

EXAMPLE 3.5. - We consider the subalgebra  $A$  of  $Free_1$  generated by the functions:

$$f(x) = \begin{cases} 3x & \text{for } 0 \leq x \leq \frac{1}{3} \\ 2 - 3x & \text{for } \frac{1}{3} \leq x \leq \frac{2}{3} \\ 3x - 2 & \text{for } \frac{2}{3} \leq x \leq 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3x & \text{for } 0 \leq x \leq \frac{1}{3} \\ 2 - 3x & \text{for } \frac{1}{3} \leq x \leq \frac{1}{2} \\ 3x - 1 & \text{for } \frac{1}{2} \leq x \leq \frac{2}{3} \\ 3x - 3 & \text{for } \frac{2}{3} \leq x \leq 1 \end{cases}.$$

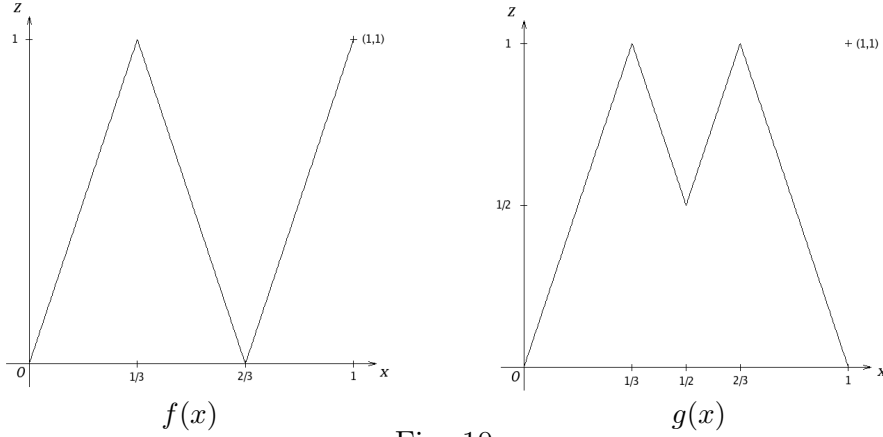


Fig. 10

We can see easily that the extremals of  $f(x)$  and  $g(x)$  are the points

$$P_0 \equiv (0, 0), \quad P_1 \equiv (1, 1), \quad P_2 \equiv (1/2, 1/2), \quad P_3 \equiv (0, 1) \quad \text{and} \quad P_4 \equiv (1, 0);$$

so that the range of  $(f, g)$  consists in the diagonals of the square  $[0, 1]^2$ .

However, these diagonals may be regarded as the set of the points of  $[0, 1]^2$  enclosed by the following four McNaughton functions

$$f_1(x) = \begin{cases} x & \text{for } 0 \leq x \leq \frac{1}{2} \\ 1 - x & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}, \quad f_2(x) = \begin{cases} 1 - x & \text{for } 0 \leq x \leq \frac{1}{2} \\ x & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases},$$

$$g_1(y) = \begin{cases} y & \text{for } 0 \leq y \leq \frac{1}{2} \\ 1 - y & \text{for } \frac{1}{2} \leq y \leq 1 \end{cases}, \quad g_2(y) = \begin{cases} 1 - y & \text{for } 0 \leq y \leq \frac{1}{2} \\ y & \text{for } \frac{1}{2} \leq y \leq 1 \end{cases},$$

that obviously satisfy the conditions  $(\bullet)$ .

It follows that the functions

$$d_1(x, y) = \begin{cases} y & \text{if } 0 \leq y \leq \frac{1}{2} \text{ and } y \geq x, \text{ or} \\ & \frac{1}{2} \leq y \leq 1 \text{ and } y \leq x, \\ 1 - y & \text{if } 0 \leq y \leq \frac{1}{2} \text{ and } y \geq 1 - x, \text{ or} \\ & \frac{1}{2} \leq y \leq 1 \text{ and } y \leq 1 - x, \\ x & \text{otherwise} \end{cases}$$

and

$$d_2(x, y) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2} \text{ and } x \geq y, \text{ or} \\ & \frac{1}{2} \leq x \leq 1 \text{ and } x \leq y, \\ 1 - x & \text{if } 0 \leq x \leq \frac{1}{2} \text{ and } x \geq 1 - y, \text{ or} \\ & \frac{1}{2} \leq x \leq 1 \text{ and } x \leq 1 - y, \\ y & \text{otherwise} \end{cases}$$

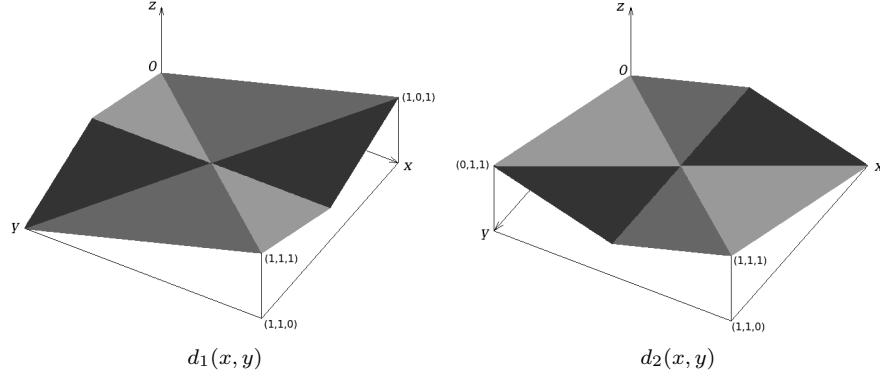


Fig. 11

are the generators of a projective subalgebra of  $Free_2$ ; but this, by the proposition 1.3, is isomorphic to  $A$ .

ii) - Now, let's consider the case that the equalizer is:

$$K = \{(x, y) \in [0, 1]^2 : g_1(y) \leq x \leq g_2(y) \text{ and } f_1(x) \leq y \leq f_2(x)\},$$

where  $f_1(x)$ ,  $f_2(x)$ ,  $g_1(y)$  and  $g_2(y)$  are broken lines, for which the conditions:

$$\begin{cases} 0 \leq f_1(x) \leq x & \text{for } 0 \leq x \leq \frac{1}{2}, \\ 0 \leq f_1(x) \leq 1 - x & \text{for } \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$\begin{cases} 1 - x \leq f_2(x) \leq 1 & \text{for } 0 \leq x \leq \frac{1}{2}, \\ x \leq f_2(x) \leq 1 & \text{for } \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$\begin{cases} 0 \leq g_1(y) \leq y & \text{for } 0 \leq y \leq \frac{1}{2}, \\ 0 \leq g_1(y) \leq 1 - y & \text{for } \frac{1}{2} \leq y \leq 1, \end{cases}$$

$$\begin{cases} 1 - y \leq g_2(y) \leq 1 & \text{for } 0 \leq y \leq \frac{1}{2}, \\ y \leq g_2(y) \leq 1 & \text{for } \frac{1}{2} \leq y \leq 1; \end{cases}$$

are satisfied,

and the sets:



$$\begin{aligned} A &= \{(x, y) \in [0, 1]^2 : y \leq f_1(x)\}, \\ B &= \{(x, y) \in [0, 1]^2 : y \geq f_2(x)\}, \\ C &= \{(x, y) \in [0, 1]^2 : x \leq g_1(y)\}, \\ D &= \{(x, y) \in [0, 1]^2 : x \geq g_2(y)\}, \end{aligned}$$

are convex.

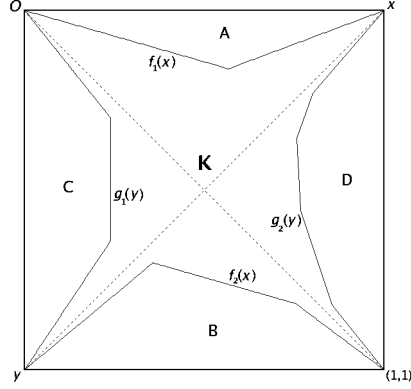


Fig. 12

For short, only a special case is examined; from which, by extension and symmetry, it follows easily how to treat the general case. Let

$$f_1(x) = \begin{cases} \frac{ax}{b} & \text{for } 0 \leq x \leq \frac{bc}{ad+bc}, \\ \frac{c(1-x)}{d} & \text{for } \frac{bc}{ad+bc} \leq x \leq 1, \end{cases}$$

with integers  $a \geq 1$ ,  $b \geq a$ ,  $c \geq 1$  and  $d \geq c$ ;

$$f_2(x) = 1, \quad g_1(y) = 0, \quad g_2(y) = 1.$$

If on the plane  $z = 0$ , we consider the points  $P \equiv \left(\frac{bc}{ad+bc}, \frac{ac}{ad+bc}\right)$

and  $Q \equiv (0, 1)$ , then the segments  $OP$  and  $PQ$  represent  $f_1(x)$ .

In the sheaves  $z - y + \lambda(by - ax) = 0$  and  $z - y + \mu(dy + cx - c)$

we should choose respectively the planes

$$z = (1 - b)y + ax \quad (1)$$

$$\text{and } z = (1 - d)y + c(1 - x), \quad (2)$$

which intersect each other along the straight line

$$\begin{cases} (d - b)y + (a + c)x = c \\ (1 - b)y + ax = z. \end{cases}$$

The plane  $z = x$  intersects the plane (1) on the straight line

$$\begin{cases} (b - 1)y = (a - 1)x \\ z = x. \end{cases}$$

On the plane  $z = 0$ , the straight lines  $(b - 1)y = (a - 1)x$

and  $(d - b)y + (a + c)x = c$  intersect each other in a point  $S \equiv (x_s, y_s)$ ,

$$\text{where } x_s = \frac{c(b - 1)}{(b - 1)(a + c) + (d - b)(a - 1)}.$$

Therefore, we must distinguish three cases, according to  $x_s \leq \frac{1}{2}$ .

If  $x_s \leq \frac{1}{2}$ , the planes  $z = 1 - x$  and  $z = x$  intersect the plane (2), respectively on the straight lines

$$\begin{cases} (d-1)y + (c-1)x = c-1 \\ z = 1-x \end{cases} \quad \text{and}$$

$$\begin{cases} (d-1)y + (c+1)x = c \\ z = x; \end{cases}$$

so that on the plane  $z = 0$ , the straight line  $(d-1)y + (c+1)x = c$  intersects the straight line  $(d-1)y + (c-1)x = c-1$

in the point  $T \equiv \left(\frac{1}{2}, \frac{1}{2} \cdot \frac{c-1}{d-1}\right)$

and obviously, the straight line  $(b-1)y + (a-1)x = 0$  in the point  $S$ . So, if  $R \equiv (0, 1/2)$  indicates the middle point of  $OQ$ , the triangle  $OPQ$  consists of two quadrilaterals  $OSTR$  and  $PSTQ$  and of two triangles  $OSP$  and  $QRT$  (Fig. 13).

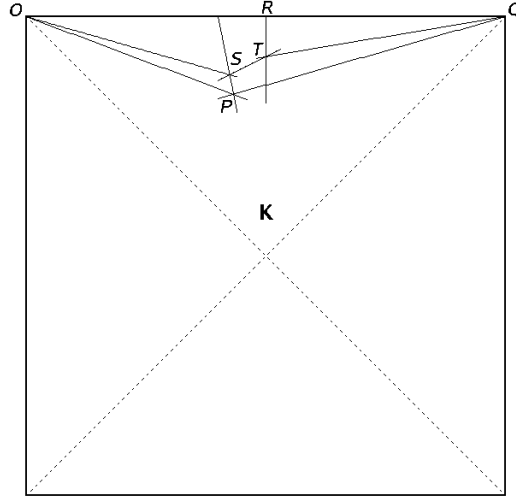


Fig. 13

And again by corollary 2.5 and proposition 3.3, it follows that the functions  $d_1(x, y) = x$  and

$$d_2(x, y) = \begin{cases} (1-b)y + ax & \text{if } (x, y) \in OSP, \\ (1-d)y + c(1-x) & \text{'' } (x, y) \in PSTQ, \\ x & \text{'' } (x, y) \in OSTR, \\ 1-x & \text{'' } (x, y) \in QRT, \\ y & \text{'' } (x, y) \in \mathbf{K} \end{cases}$$

generate a projective subalgebra of  $Free_2$ .

If  $x_s = \frac{1}{2}$ , then the points  $S$  and  $T$  coincide; so that, for the definition of  $d_2(x, y)$ , the four triangles  $OSP$   $PSQ$   $OSR$  and  $QRS$  are to be considered (Fig. 14).

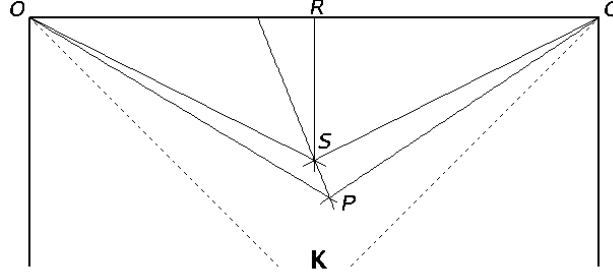


Fig. 14

That is

$$d_2(x, y) = \begin{cases} (1-b)y + ax & \text{if } (x, y) \in OSP, \\ (1-d)y + c(1-x) & \text{'' } (x, y) \in PSQ, \\ x & \text{'' } (x, y) \in OSR, \\ 1-x & \text{'' } (x, y) \in QRS, \\ y & \text{'' } (x, y) \in \mathbf{K}. \end{cases}$$

Finally, if  $x_s \geq \frac{1}{2}$ , then on the plane  $z = 0$ , the straight lines

$(d-1)y + (c-1)x = c-1$  and  $(d-b)y + (a+c)x = c$  intersect each other in a point  $U \equiv (x_U, y_U)$  where  $x_U = \frac{c(b-1) + d-b}{(d-1)(a+c) - (d-b)(c-1)}$ ;

it is easy to check that  $x_s \geq \frac{1}{2}$  implies  $x_U \geq \frac{1}{2}$ .

On the plane  $z = 0$ , the straight line  $(b-1)y - (a+1)x = -1$ , projection of the straight line intersection of the plane (1) with the plane  $z = 1-x$ , intersects the straight line  $(b-1)y - (a-1)x = 0$  at the point  $V \equiv \left(\frac{1}{2}, \frac{1}{2} \cdot \frac{a-1}{b-1}\right)$  and the straight line  $(d-1)y + (c-1)x = c-1$  at the point  $U$  (Fig. 15).

We have:

$$d_2(x, y) = \begin{cases} (1-b)y + ax & \text{if } (x, y) \in OVUP, \\ (1-d)y + c(1-x) & \text{'' } (x, y) \in PUQ, \\ x & \text{'' } (x, y) \in OVR, \\ 1-x & \text{'' } (x, y) \in QUVR, \\ y & \text{'' } (x, y) \in \mathbf{K}. \end{cases}$$

$$f_1(x) = \begin{cases} \frac{2x}{7} & \text{for } 0 \leq x \leq \frac{21}{37}, \\ \frac{3(1-x)}{8} & \text{for } \frac{21}{37} \leq x \leq 1, \end{cases}$$

$$f_2(x) = 1, \quad g_1(y) = 0, \quad g_2(y) = 1.$$

Fig. 15

$$d_1(x, y) = x,$$

$$d_2(x, y) = \begin{cases} 2(x - 3y) & \text{if } y \leq 2x/7 \text{ and } y \leq 3 - 5x \\ & \text{and } y \geq x/6 \text{ and } y \geq (3x - 1)/6, \\ 3(1 - x) - 7y & \text{" } y \geq 3 - 5x \text{ and } y \leq 3(1 - x)/8 \\ & \text{and } y \geq 2(1 - x)/7, \\ x & \text{" } y \leq x/6 \text{ and } x \leq 1/2, \\ 1 - x & \text{" } x \geq 1/2 \text{ and } y \leq (3x - 1)/6 \\ & \text{and } y \leq 2(1 - x)/7, \\ y & \text{" } y \geq 2x/7 \text{ or } y \geq 3(1 - x)/8. \end{cases}$$

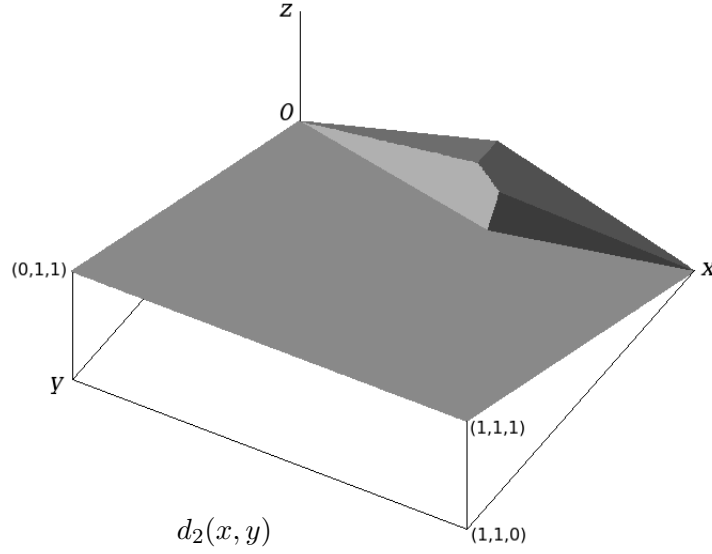


Fig. 16

iii) - Now we examine the case that equalizer  $K$  is identified by a generic triangle with a vertex at the origin.

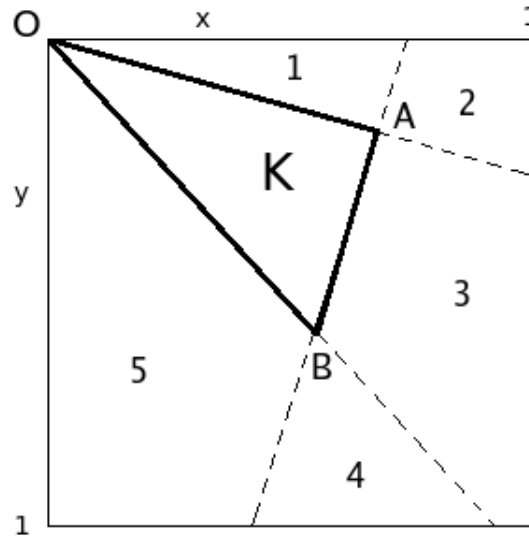


Fig. 17

Let  $ax + by = 0$ ,  $a_1x + b_1y = 0$  and  $a_2x + b_2y + c = 0$  be respectively the equations of the straight lines for  $OA$ ,  $OB$ ,  $AB$  (Fig. 17), with  $a, a_1, a_2, b, b_1, b_2 \in \mathbb{Z}$ ,  $a, a_1, c > 0$ ,  $b, b_1 < 0$ , and  $(a, b) = (a_1, b_1) = (a_2, b_2, c) = 1$ . Then:

$$K = \{(x, y) : ax + by \leq 0 \text{ and } a_1x + b_1y \geq 0 \text{ and } a_2x + b_2y + c \geq 0\}.$$

We can apply theorem 2.4, taking into account the proposition 3.3; more precisely, we proceed to the construction of the generators  $d_1$  and  $d_2$  assuming that they coincide, respectively, with  $g_1 = x$  and  $g_2 = y$  on the equalizer  $K$ . Then, we connect these “basic pieces” of  $d_1$  and  $d_2$  with the plane  $z = 0$ , by appropriate planes, so that the hypothesis of the theorem 2.4 are satisfied.

For the side  $OA$ , we consider two planes belonging respectively to the sheaves  $z = x + k(ax + by)$  and  $z = y + h(ax + by)$ . As the only point of  $K$ , where one of the coordinates  $x, y$  takes zero value is the origin of axes, then these two planes must intersect necessarily the plane  $z = 0$  on the same straight line. This implies

$$(ka + 1)(hb + 1) - hkb = 0,$$

that is

$$ka + hb + 1 = 0.$$

So, the planes of the relative sheaves that satisfy such condition are:

$$z = -hbx + kby \quad (1)$$

$$z = hax - kay \quad (2)$$

and their intersection straight line with the plane  $z = 0$  is

$$y = \frac{h}{k}x. \quad (3)$$

If  $k_0, h_0$  is a particular solution of the diophantine equation  $ak + bh = -1$ , the general solution has the form:

$$k = k_0 - sb, \quad h = h_0 + sa, \quad \text{with } s \in \mathbb{Z} \text{ wichever;}$$

so that  $\frac{h}{k} = -\frac{a}{b} + \frac{1/b^2}{s - k_0/b}$  and for  $s < \frac{k_0}{b} + \frac{1}{ab}$  we have:  $0 < \frac{h}{k} < -\frac{a}{b}$ .

In a similar manner, we proceed for the side  $OB$ ; fixing the planes

$$z = -h'b_1x + k'b_1y \quad (4)$$

$$z = h'a_1x - k'a_1y \quad (5)$$

as well as their intersection straight line with the plane  $z = 0$

$$y = \frac{h'}{k'}x \quad (6)$$

with the conditions  $k'a_1 + h'b_1 + 1 = 0$  and  $\frac{h'}{k'} > -\frac{a_1}{b_1}$ .

Now, we indicate with  $A'$  and  $A''$ , respectively, the intersection points of the vertical for  $A$  with the planes  $z = x$  and  $z = y$ ; with  $B'$  and  $B''$  the intersection points of the vertical for  $B$  with the same planes.

Set  $\delta = ba_2 - ab_2$  and  $\delta_1 = b_1a_2 - a_1b_2$ , we have:

$$A \equiv \left( \frac{-bc}{\delta}, \frac{ac}{\delta} \right) \quad \text{and} \quad B \equiv \left( \frac{-b_1c}{\delta_1}, \frac{a_1c}{\delta_1} \right),$$

with  $\delta, \delta_1 > 0$ ; consequently

$$A' \equiv \left( \frac{-bc}{\delta}, \frac{ac}{\delta}, \frac{-bc}{\delta} \right), \quad A'' \equiv \left( \frac{-bc}{\delta}, \frac{ac}{\delta}, \frac{ac}{\delta} \right) \quad \text{and}$$

$$B' \equiv \left( \frac{-b_1c}{\delta_1}, \frac{a_1c}{\delta_1}, \frac{-b_1c}{\delta_1} \right), \quad B'' \equiv \left( \frac{-b_1c}{\delta_1}, \frac{a_1c}{\delta_1}, \frac{a_1c}{\delta_1} \right).$$

We consider the sheaves of planes

$$z - x + l(a_2x + b_2y + c) = 0 \quad \text{and} \quad z - y + m(a_2x + b_2y + c) = 0,$$

generated by the plane  $a_2x + b_2y + c = 0$  and, respectively, by the planes  $z = x$  and  $z = y$ ; these sheaves intercept on the plane  $z = 0$  the sheaves of straight lines

$$x = l(a_2x + b_2y + c) \quad \text{and} \quad y = m(a_2x + b_2y + c).$$

So the coordinates of the point  $P$ , intersection of a straight line of the first sheaf with one straight line of the second, must satisfy the system

$$\begin{cases} l(a_2x + b_2y + c) = x \\ m(a_2x + b_2y + c) = y. \end{cases}$$

It follows, immediately, that the point  $P$  must lie on the straight line through the origin of equation  $mx = ly$  and that

$$P \equiv \left( \frac{lc}{1 - la_2 - mb_2}, \frac{mc}{1 - la_2 - mb_2} \right).$$

In order that, the straight line through  $OP$  lies between the straight lines through  $OA$  and  $OB$  (Fig. 18), it must be  $-\frac{a}{b} < \frac{m}{l} < -\frac{a_1}{b_1}$ , and the parameters  $m$  and  $l$  must have the same sign.

If  $m < 0$  and  $l < 0$ , must be also  $1 - la_2 - mb_2 < 0$ ; so that is

$$a_2 \frac{lc}{1 - la_2 - mb_2} + b_2 \frac{mc}{1 - la_2 - mb_2} + c = \frac{c}{1 - la_2 - mb_2} < 0,$$

therefore  $P$  is an outer point to the equalizer  $K$  (it is in zone 3) and when  $m$  and  $l$  in absolute value increase, its distance from the side  $AB$  of  $K$  decreases.

The equation of a generic straight line of the sheaf generated by a straight

line of the first sheaf with one of the second, (that is a straight line of the plane  $z = 0$  passing through  $P$ ) is:

$$h[l(a_2x + b_2y + c) - x] + k[m(a_2x + b_2y + c) - y] = 0,$$

with  $h, k \in \mathbb{Z}$ , wich can be put in the form

$$[a_2(hl + km) - h]x + [b_2(hl + km) - k]y + c(hl + km) = 0.$$

Let

$$\bar{a}x + \bar{b}y + \bar{c} = 0 \quad (7)$$

$$\text{and} \quad \hat{a}x + \hat{b}y + \hat{c} = 0 \quad (8)$$

be the equations for two straight lines belonging to this sheaf.

The equation of a McNaughton generic plane of the sheaf whose axis is the straight line (7), can be written in the form

$$z = \lambda(\bar{a}x + \bar{b}y + \bar{c}) = \lambda\left\{\bar{h}[l(a_2x + b_2y + c) - x] + \bar{k}[m(a_2x + b_2y + c) - y]\right\},$$

with  $\lambda$  integer, and in order that the point  $A'$  lies on this plane, it must be:

$$-\frac{bc}{\delta} = \lambda'\left\{\bar{h}\left[\frac{bc}{\delta}\right] + \bar{k}\left[-\frac{ac}{\delta}\right]\right\}, \quad \text{from wich} \quad -b = \lambda'(\bar{h}b - \bar{k}a);$$

while, if we require that the point  $A''$  lies on a plane of the same sheaf, it must be:

$$\frac{ac}{\delta} = \lambda''\left\{\bar{h}\left[\frac{bc}{\delta}\right] + \bar{k}\left[-\frac{ac}{\delta}\right]\right\}, \quad \text{from wich} \quad a = \lambda''(\bar{h}b - \bar{k}a).$$

As  $(a, b) = 1$ , it follows necessarily

$$\bar{h}b - \bar{k}a = \mp 1, \quad \lambda' = \pm b \quad \text{and} \quad \lambda'' = \mp a.$$

If we choose  $\bar{h}b - \bar{k}a = 1$ , then  $\lambda' = -b$  and  $\lambda'' = a$  and the equations of two planes of the sheaf, one passing through  $A'$ , the other through  $A''$ , are respectively:

$$z = -b\bar{a}x - b\bar{b}y - b\bar{c} \quad (9)$$

$$\text{and} \quad z = a\bar{a}x + a\bar{b}y + a\bar{c}. \quad (10)$$

These planes intersect  $z$ -axis in two points with

$$z' = -b\bar{c} = -bc(\bar{h}l + \bar{k}m) \quad \text{and} \quad z'' = a\bar{c} = ac(\bar{h}l + \bar{k}m);$$

so that  $z'$  and  $z'' > 0$  if and only if  $\bar{h}l + \bar{k}m > 0$ .

Now the equation  $\bar{h}b - \bar{k}a = 1$  admits the integer solutions:

$\bar{h} = h_1 + at$ ,  $\bar{k} = k_1 + bt$ , with arbitrary integer parameter  $t$ ; so the condition  $\bar{h}l + \bar{k}m > 0$  is equivalent to  $t(la + mb) > -mk_1 - lh_1$  and this is satisfied

for  $t > -\frac{mk_1 + lh_1}{la + mb}$  (being  $la + mb > 0$ ).



Similarly, the equations of the two planes of the sheaf with the straight line (8) as axis, passing respectively through the points  $B'$  and  $B''$ , are:

$$z = -b_1 \hat{a}x - b_1 \hat{b}y - b_1 \hat{c} \quad (11)$$

$$\text{and} \quad z = a_1 \hat{a}x + a_1 \hat{b}y + a_1 \hat{c}; \quad (12)$$

with the integer parameters  $\hat{h}$  and  $\hat{k}$  (wich determine  $\hat{a}$ ,  $\hat{b}$  and  $\hat{c}$ ) satisfying the conditions  $\hat{h}b_1 - \hat{k}a_1 = 1$ , and  $\hat{h}l + \hat{k}m > 0$ .

Finally, we note that the planes through the points  $A'$ ,  $B'$ ,  $P$  and  $A''$ ,  $B''$ ,  $P$  have, respectively, equations:

$$z = (1 - la_2)x - lb_2y - lc \quad (13)$$

$$\text{and} \quad z = -ma_2x + (1 - mb_2)y - mc. \quad (14)$$

We denote, also respectively, by  $Q$  and  $R$  the intersection points between the straight lines (3), (7) and (6), (8).

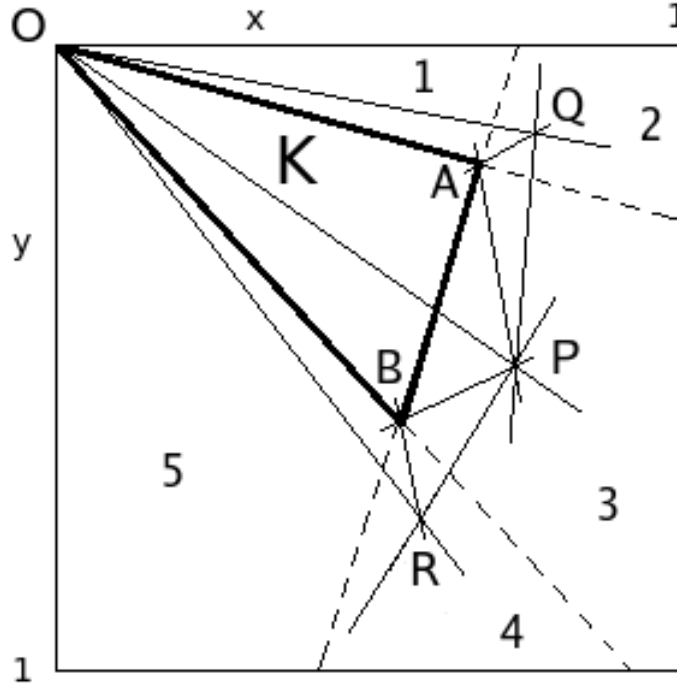


Fig. 18

So the generators  $d_1$  and  $d_2$ ,

zero along and outside of the broken line  $O Q P R O$ , coincide:

$d_1$  with planes (1), (9), (13), (11), (4) and  $z = x$

and  $d_2$  with planes (2), (10), (14), (12), (5) and  $z = y$ ,

respectively, on the triangles

$O Q A$ ,  $Q A P$ ,  $A P B$ ,  $P B R$ ,  $B R O$  and  $K$  (Fig. 18).

It follows immediately, by proposition 3.3, that the generators  $d_1$  and  $d_2$ ,

so defined, satisfy the conditions of the theorem 2.4.

These considerations can be easily extended also to the case that equalizer  $K$  is decomposable into triangles, all with a vertex in the origin. Infact, if the construction, above described, is repeated for each triangle

$$O A_1 B_1, O A_2 B_2, \dots, O A_n B_n,$$

we obtain in particular respectively the quadrilaterals

$$O Q_1 P_1 R_1, O Q_2 P_2 R_2, \dots, O Q_n P_n R_n,$$

whose union is a polygon

$$O Q_1 P_1 S_1 \dots S_m P_n R_n O$$

that contains  $K$ . As each segment of the perimeter of this polygon belongs to one of the quadrilaterals  $O Q_i P_i R_i$ , it is possible to use convenient pieces of planes determinated for each triangle, to define the generators (Fig. 19).

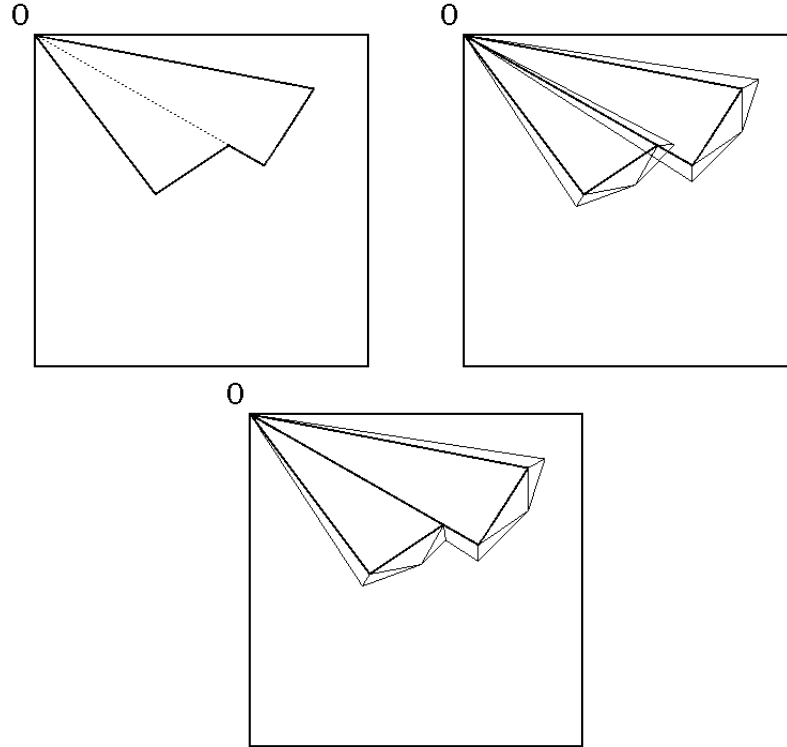


Fig. 19

In conclusion, using the methods described in § 3 b), in the case of equalizers containing the diagonals and whose boundary consists of continuous and piecewise linear functions, projective generators are obtained. Similarly, the same result is obtained in § 3 c), in the case of equalizers formed by triangles with a common vertex coinciding with one of the vertices of  $[0, 1]^2$ . Apart

from the complication in the calculations, the results can be extended to the case of three or more generators. It also seems likely that the methods described in § 3 can be applied in other cases.

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